

Restricted involutions and Motzkin paths

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Abstract. We show how a bijection due to Biane between involutions and labelled Motzkin paths yields bijections between Motzkin paths and two families of restricted involutions that are counted by Motzkin numbers, namely, involutions avoiding 4321 and 3412. As a consequence, we derive characterizations of Motzkin paths corresponding to involutions avoiding either 4321 or 3412 together with any pattern of length 3. Furthermore, we exploit the described bijection to study some notable subsets of the set of restricted involutions, namely, fixed point free and centrosymmetric restricted involutions.

Keywords: permutations with restricted patterns, involutions, Motzkin paths.

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1 Introduction

A permutation $\sigma \in S_n$ avoids the pattern $\tau \in S_k$ if σ does not contain a subsequence order-isomorphic to τ . Permutations with forbidden patterns have been intensively studied in recent years for their connection with many problems arising in both computer science and combinatorics, such as Schubert varieties, Kazdan-Lusztig polynomials, Chebyshev polynomials and various sorting algorithms (see [10], [13], [14], [16], and references there in). Many classical sequences, such as Catalan numbers, Motzkin numbers, central binomial coefficients and Fibonacci numbers, occur when the cardinalities of sets of pattern avoiding permutations are computed.

In particular, Motzkin numbers arise mainly when involutions avoiding certain patterns of length 4 are studied. More precisely, involutions on n objects

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avoiding the patterns 3412, 4321, 2143, and 1234 enumerated by Motzkin numbers (see [9], [12], and [15]).

On the other hand, in the literature there are many bijections between permutations and Motzkin paths provided with some kind of labelling (see [1], [4], [7], and [8]). In this paper, we consider the bijection defined by Biane in [1], restricted to the set of involutions. The bijection maps an involution into a Motzkin path whose down steps are labelled with an integer that does not exceed its height, while the other steps are unlabelled.

This bijection reveals to be an effective tool for characterizing several sets of restricted involutions, and allows us both to recover already known results in a unified framework, and to obtain some new characterizations.

In particular, we prove that an involution τ avoids 4321 if and only if the label of any down step in the corresponding Motzkin path is 1. Similarly, an involution τ avoids 3412 if and only if the label of any down step in the corresponding Motzkin path equals its height.

These two results allow us, on the one hand, to regain the two standard bijections between (unlabelled) Motzkin paths and involutions avoiding either 4321 or 3412 (see [9]), and, on the other hand, to characterize those labelled Motzkin paths that correspond to involutions avoiding one of these two patterns together with an assigned pattern of length 3, hence obtaining some new enumerative results.

In the last two sections, as an example of the efficacy of the present approach, we study pattern avoidance on sets of involutions corresponding to two notable subsets of Motzkin paths, namely, Dyck paths and Motzkin paths that are symmetric with respect to a vertical line.

2 Preliminary notions

Given a permutation $\sigma \in S_n$, one can partition the set $\{1, 2, \dots, n\}$ into intervals A_1, \dots, A_t (i.e. $A_i = \{k, k+1, \dots, k+h\}$) such that $\sigma(A_j) = A_j$ for every j . The restrictions of σ to the intervals in the finest of these decompositions are called *connected components* of σ . A permutation σ with a single connected component is called *connected*.

For example, the permutation

$$\sigma = 2 \ 4 \ 5 \ 3 \ 1 \ 7 \ 6$$

has the two connected components $\sigma_1 = 2\ 4\ 5\ 3\ 1$ and $\sigma_2 = 7\ 6$, while the permutation

$$\rho = 2\ 7\ 6\ 1\ 3\ 5\ 4$$

is connected.

Let $\psi \in S_n$ be defined by $\psi(i) = n + 1 - i$. If $\sigma \in S_n$, the *reverse* of σ is the permutation $\sigma_r = \sigma\psi$. Similarly, the *complement* of σ is the permutation $\sigma_c = \psi\sigma$ and the *reverse-complement* of σ is the permutation $\sigma_{rc} = \psi\sigma\psi$.

Let $\sigma \in S_n$ and $\pi \in S_k$, $k \leq n$, be two permutations. We say that σ *contains the pattern* π if there exists a subsequence $(\sigma(i_1), \sigma(i_2), \dots, \sigma(i_k))$ with $1 \leq i_1 < i_2 < \dots < i_k \leq n$ that is order-isomorphic to π . We say that σ *avoids* the pattern π if σ does not contain π . Given a set of permutations $A \subseteq S_n$ and a set of patterns π_1, \dots, π_k , we denote by $A(\pi_1, \dots, \pi_k)$ the set of elements in A that avoid π_i for every $1 \leq i \leq k$.

We say that two patterns π_1 and π_2 are *equidistributed* over a set of permutations A if $|A(\pi_1)| = |A(\pi_2)|$ and that they are *equivalent* over A if $A(\pi_1) = A(\pi_2)$.

Recall that a permutation $\tau \in S_n$ is an involution if and only if $\tau^{-1} = \tau$. Equivalently, τ is an involution if and only if its cycle structure has no cycle of length longer than two.

Denote by I_n the set of involutions in the symmetric group S_n . We point out that the bijection that associates a permutation σ to its reverse-complement σ_{rc} maps I_n into itself. This implies that a pattern $\pi \in S_k$ and its reverse-complement π_{rc} are equidistributed over I_n . In fact, a permutation σ contains π if and only if σ_{rc} contains π_{rc} .

Moreover, any pattern π is equivalent over I_n to its inverse π^{-1} , since, in general, for every permutation $\sigma \in S_n$, σ contains π if and only if σ^{-1} contains π^{-1} .

3 Labelled Motzkin paths and involutions

A *Motzkin path* of length n is a lattice path starting at $(0, 0)$, ending at $(n, 0)$, and never going below the x -axis, consisting of up steps $U = (1, 1)$, horizontal steps $H = (1, 0)$, and down steps $D = (1, -1)$. A *Dyck path* is a Motzkin path that does not contain horizontal steps.

The set of Motzkin paths of length n will be denoted by \mathcal{M}_n , while the set of Dyck paths of length $2n$ will be denoted by \mathcal{D}_n . It is well known that the cardinality of \mathcal{M}_n is the n -th *Motzkin number* M_n (sequence A001006 in [18]), and that the cardinality of the set \mathcal{D}_n is the n -th Catalan number C_n (sequence A000108 in [18]).

We define the *height of a step* of a Motzkin path to be the larger y coordinate of the step, and the *height of a path* to be the largest height of its steps.

An *irreducible* (or *elevated*) Motzkin path is a Motzkin path that does not touch the x -axis except for the origin and the final destination. An *irreducible component* of a Motzkin path M is a maximal irreducible Motzkin subpath of M .

A *labelling* of a Motzkin path M will be a map that associates with every down step D at height h an integer $\lambda(D)$ such that $1 \leq \lambda(D) \leq h$. A *labelled Motzkin path* is a pair (M, λ) , where M is a Motzkin path and λ a labelling of M .

We will denote by v the *unitary labelling*, namely, the labelling that assigns the label 1 to every down step D of M . Similarly, the *maximal labelling* μ will be defined by $\mu(D) = h(D)$, where $h(D)$ is the height of D .

We now define a bijection Φ between the set I_n of involutions of the symmetric group S_n and the set of labelled Motzkin paths of length n , that is essentially a restriction to the set of involution of the bijection appearing in [1].

Consider an involution τ and determine the set $\text{exc}(\tau) = \{i | \tau(i) > i\}$ of its *excedances*. Starting from the empty path and from the list $A_\tau = (a_1 \ a_2 \ \dots \ a_s)$ consisting of the elements of $\text{exc}(\tau)$ written in increasing order, we construct the labelled Motzkin path $\Phi(\tau)$ by adding a step for every integer $1 \leq i \leq n$ as follows:

- if $\tau(i) = i$ we add a horizontal step at i -th position;
- if $\tau(i) > i$ we add an up step at i -th position;
- if $\tau(i) < i$ we add a down step at i -th position. The integer $\tau(i)$ appears in the list A_τ at position k , say. We assign the label k to the new down step and remove $\tau(i)$ from A_τ .

For example, the image $\Phi(\tau)$ of the involution $\tau = 4\ 7\ 5\ 1\ 3\ 6\ 2\ 9\ 8$ is the labelled Motzkin path in Figure 1.

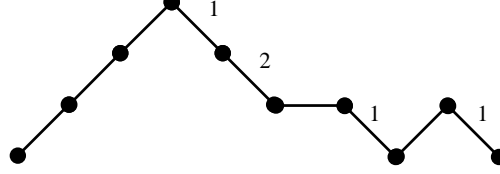


Figure 1: The labelled Motzkin path $\Phi(\tau)$, with $\tau = 4\ 7\ 5\ 1\ 3\ 6\ 2\ 9\ 8$

Note that an involution τ is connected if and only if its image is an irreducible Motzkin path.

There is a simple relationship between the Motzkin paths associated with an involution τ and its reverse-complement τ_{rc} :

Theorem 1 *Let (M, λ) and (M', λ') the labelled Motzkin paths of length n associated by Φ with τ and τ_{rc} , respectively. Then, the Motzkin paths M and M' have the following symmetry (with respect to the vertical line $x = \frac{n}{2}$):*

- i. if M has a horizontal step at the k -th position, then M' has a horizontal step at the $(n + 1 - k)$ -th position;*
- ii. if M has an up step at the k -th position, then M' has a down step at the $(n + 1 - k)$ -th position;*
- iii. if M has a down step at the k -th position, then M' has an up step at the $(n + 1 - k)$ -th position.*

Proof Case *i.* is easily verified. For case *ii.*, suppose that M has an up step at the k -th position. This implies that $j = \tau(k) > k$. Then we must have $\tau_{rc}(n + 1 - k) = n + 1 - j < n + 1 - k$, and hence the $n + 1 - k$ -th step in M' is a down step. The last case follows analogously.

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Motzkin numbers appear in the enumeration of involutions avoiding several patterns of length 4. More precisely, for τ equal to 1234, 1243, 3412, 3214,

1432, 4321 or 2143, $|I_n(\tau)| = M_n$ (see [2], [5], [9], [12], and [15]). Hence, it is natural to search for a bijection between the sets $I_n(\tau)$ and \mathcal{M}_n for each one of these patterns. Some of these bijections appear in the literature (see e.g. [2] and [5]). We will show that the described correspondence between I_n and the set of labelled Motzkin paths of length n provides in a natural way such a bijection in the cases $\tau = 3412$ and $\tau = 4321$.

4 4321-avoiding involutions

Theorem 2 *Let τ be an involution and $\Phi(\tau) = (M, \lambda)$. Then, τ avoids the pattern 4321 if and only if λ is the unitary labelling.*

Proof Suppose that there exists a down step at position i in M with a label greater than 1. Clearly, such a step can not be the last one in M . Let j be the least position greater than i corresponding to a down step whose label is 1 (in the worst case, $j = n$). Then, τ contains the subsequence

$$j \ i \ \tau(i) \ \tau(j)$$

which is of type 4321.

Conversely, suppose that each label in M equals 1. In this case, τ is obtained by interlacing the three following increasing subsequences:

- the sequence of fixed points;
- the sequence of excedances;
- the sequence of deficiencies, namely, the sequence of the images of excedances of τ ,

and, hence, τ can not contain any decreasing subsequence of length 4.

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We remark that also the involutions in $I_n(321)$ and in $I_n(312)$ have a simple characterization in term of labelled Motzkin paths:

Proposition 3 *Let τ be an involution in I_n and $\Phi(\tau) = (M, \lambda)$. Then, τ avoids 321 if and only if $\lambda = v$ and all horizontal steps in M are at height 0.*

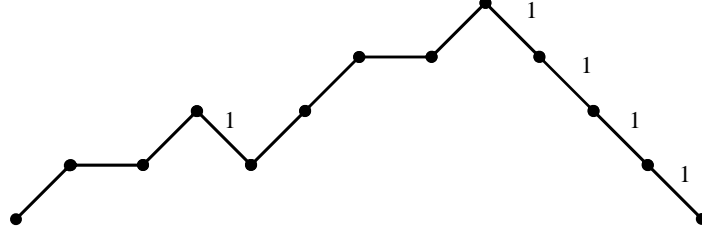


Figure 2: The labelled Motzkin path associated with the 4321 avoiding involution $\tau = 4\ 2\ 9\ 1\ 10\ 11\ 7\ 12\ 3\ 5\ 6\ 8$

Proof Suppose that M contains a horizontal step H at position k and height $h > 0$. In this case, consider the up step U preceding H and closest to H . Denote by j the position of U in M . The integer j corresponds to the position of an excedance of τ , so the involution τ contains the 321-sequence $\tau(j)\ k\ j$.

Suppose now that the labelling λ is not unitary. As seen above, in this case τ contains a 4321-subsequence, and hence contains also the pattern 321.

Conversely, suppose that M has no horizontal steps at a height greater than 0 and that λ is unitary. Then, the irreducible components of M are either horizontal steps at height 0 or Dyck paths. Recall that the irreducible components of M correspond to the connected components of τ . The connected components corresponding to horizontal steps avoid trivially the pattern 321. On the other hand, each component corresponding to a Dyck path (with the unitary labelling) can be obtained by interlacing the increasing sequences of its excedances and deficiencies. This completes the proof.

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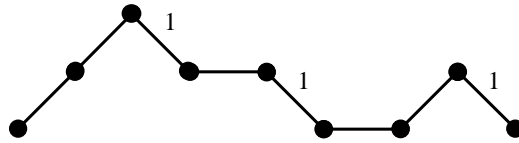


Figure 3: The labelled Motzkin path corresponding to $\tau = 3\ 5\ 1\ 4\ 2\ 6\ 8\ 7$, that contains 321.

Proposition 4 *Let τ be an involution in I_n and (M, λ) the associated labelled Motzkin path. Then, τ avoids 312 if and only if $\lambda = \mu$ and the irreducible components of M are of type either $U^a H D^a$ or $U^a D^a$.*

Proof Remark that an involution τ avoids 312 if and only if there exist integers $k_1 < k_2 < \dots < k_p$ such that τ can be written in one-line notation as follows

$$k_1 \ k_1 - 1 \ \dots \ 1 \ k_2 \ k_2 - 1 \ \dots \ k_1 + 1 \ \dots \ k_p \ \dots \ k_{p-1} + 1.$$

In fact, suppose that $\tau(1) = k_1$. Hence $\tau(k_1) = 1$, since τ is an involution. As τ avoids 312, we must have

$$\tau(2) = k_1 - 1, \ \tau(3) = k_1 - 2, \ \dots \ \tau(k_1 - 1) = 2.$$

Iterating this argument we get the assertion.

This implies that involutions in $I_n(312)$ correspond via the bijection Φ to Motzkin paths whose irreducible components are of type either $U^a H D^a$ or $U^a D^a$, with the maximal labelling.

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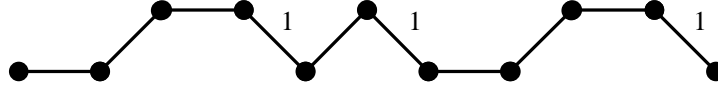


Figure 4: The labelled Motzkin path corresponding to $\tau = 1 \ 4 \ 3 \ 2 \ 6 \ 5 \ 7 \ 10 \ 9 \ 8$, that avoids 312.

We now want to characterize labelled Motzkin paths corresponding to involutions avoiding 4321 and a pattern σ of length 3. We do not consider the case $\sigma = 123$ since, for every $n > 6$, $I_n(4321, 123) = \emptyset$.

Theorem 5 *Let τ be an involution in $I_n(4321)$ associated with the labelled Motzkin path (M, v) . Then:*

- i. τ avoids 132 if and only if M does not contain any subpath among HU , DU and DHD . As a consequence, τ avoids 213 if and only if M does not contain any subpath among DH , DU and UHU ;*

- ii. τ avoids 321 if and only if all horizontal steps in M have height 0;
- iii. τ avoids 312, and hence 231, if and only if the irreducible components of M are either H , or UHD , or UD .

Proof

- i. Consider the pattern 132. Let (M, v) be a Motzkin path with the unitary labelling. If M contains one subpath among HU , DU and DHD , then the corresponding involution τ must contain 132. In fact:
 - 1. if the k -th step in M is horizontal and the $(k+1)$ -th step is an up step, then $\tau(k) = k$ and $\tau(k+1) = j > k+1$. Hence, τ contains the 132-pattern $k \ j \ k+1$;
 - 2. if the k -th step in M is a down step and the $(k+1)$ -th step is an up step, then $\tau(k) = j < k$ and $\tau(k+1) = m > k+1$. Hence, τ contains the 132-pattern $j \ m \ k+1$;
 - 3. if the k -th step in M is a down step, the $(k+1)$ -th step is horizontal and the $(k+2)$ -th step is a down step, then $\tau(k) = j < k$, $\tau(k+1) = k+1$ and $\tau(k+2) = m$, with $j < m < k$. Hence, τ contains the 132-pattern $j \ k+1 \ m$.

If M does not contain any of those subpaths, then $M = U^a H^b D^a H^c$ for suited non-negative integers a, b, c such that $2a + b + c = n$. A Motzkin path of this type corresponds to an involution τ whose one-line notations is

$$e_1 \ \cdots \ e_a \ a+1 \ \cdots \ a+b \ d_1 \ \cdots \ d_b \ 2a+b+1 \ \cdots \ n,$$

where $e_1 \dots e_a$ are the excedances of τ written in increasing order and $d_1 \dots d_b$ are the deficiencies of τ in increasing order. In this case, τ clearly avoids 132.

Remark now that the pattern 213 is the reverse-complement of 132. As in the case of the whole set I_n , the set $I_n(4321)$ is closed under reverse-complement. In fact, let $\tau \in I_n(4321)$ and suppose that $\sigma = \tau_{rc}$ contains the pattern 4321. Then, there exist integers $i < j < h < k$ such that $\sigma(i) > \sigma(j) > \sigma(h) > \sigma(k)$. Since $\sigma(x) = n+1 - \tau(n+1-x)$, the involution τ contains the subsequence $n+1-k > n+1-h > n+1-j > n+1-i$, and this yields a contradiction. Then, if τ

corresponds to the labelled Motzkin path (M, v) , then τ_{rc} corresponds to the path M' obtained by reflecting M over the line $x = \frac{n}{2}$ (see Theorem 1), with the unitary labelling.

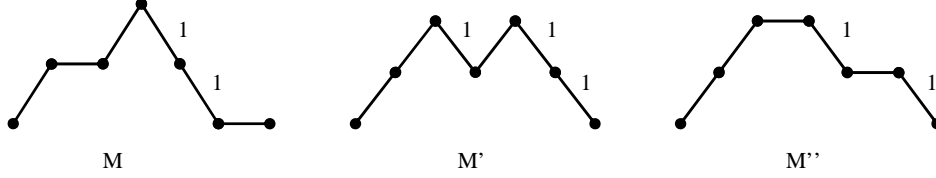


Figure 5: The labelled Motzkin paths M , M' , and M'' corresponding to the involutions $\tau = 4 \mathbf{2} \mathbf{5} \mathbf{1} \mathbf{3} \mathbf{6}$, $\tau' = 3 \mathbf{5} \mathbf{1} \mathbf{6} \mathbf{2} \mathbf{4}$, $\tau'' = 4 \mathbf{6} \mathbf{3} \mathbf{1} \mathbf{5} \mathbf{2}$, all containing 132.

- ii. The assertion follows immediately from Proposition 3, since an involution that avoids 321 avoids also 4321.
- iii. Recall that involutions in $I_n(312)$ correspond to Motzkin paths whose irreducible components are either $B_a = U^a H D^a$ or $P_a = U^a D^a$, with the maximal labelling (see Proposition 4). Such a path corresponds to an involution in $I_n(4321)$ if and only if its height is 1, namely, it is a concatenation of subpaths of the kind $B_0 = H$, $B_1 = U H D$ and $P_1 = U D$. This completes the proof.

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The preceding theorem yields the following enumerative results. Formulas *i.* and *iii.*, up to our knowledge, are new.

Corollary 6 *We have:*

- i.* $|I_n(4321, 132)| = |I_n(4321, 213)| = 1 + \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$;
- ii.* $|I_n(4321, 321)| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$;
- iii.* $I_n(4321, 312) = I_n(4321, 231)$ and $|I_n(4321, 312)| = t_{n+2}$, where t_n is the sequence of Tribonacci numbers defined by $t_0 = 0$, $t_1 = 0$, $t_2 = 1$ and $t_{n+3} = t_{n+2} + t_{n+1} + t_n$.

Proof

- i. The cardinality of $I_n(4321, 132)$ equals the number of Motzkin paths of the kind $M = U^a H^b D^a H^c$. Hence, we have:

$$|I_n(4321, 132)| = 1 + \sum_{h=1}^{\lfloor \frac{n}{2} \rfloor} n - 2h + 1 = 1 + \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil.$$

- ii. Obviously $I_n(4321, 321) = I_n(321)$. It is well known (see [17]) that the cardinality of $I_n(321)$ equals the n -th central binomial coefficient.
- iii. A Motzkin path associated with an involution in $I_n(4321, 312)$ has irreducible components of the kind H , $B_1 = UH D$ and $P_1 = U D$. Hence, such a Motzkin path $M \in \mathcal{M}_{n+3}$ can be obtained by adding
1. an H step at the end of a path of the same kind in \mathcal{M}_{n+2} ;
 2. a subpath P_1 at the end of a path of the same kind in \mathcal{M}_{n+1} ;
 3. a subpath B_1 at the end of a path of the same kind in \mathcal{M}_n .

Moreover, $|I_1(4321, 312)| = 1 = t_3$, $|I_2(4321, 312)| = 2 = t_4$ and $|I_3(4321, 312)| = 4 = t_5$. This completes the proof.

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5 3412-avoiding involutions

Also 3412-avoiding involutions are enumerated by the Motzkin numbers (see [9]). The labelled Motzkin paths associated with an involution in $I_n(3412)$ can be characterized as follows:

Theorem 7 *Let τ be an involution associated with the labelled Motzkin path (M, λ) . Then, τ avoids the pattern 3412 if and only if λ is the maximal labelling.*

Proof Let τ be the involution corresponding to a given labelled Motzkin path and suppose that there exists a down step D at position k whose label is not maximal. This means that, at the k -th step of the procedure that creates the Motzkin path $\Phi(\tau)$, the maximal element a in the list A_τ is not removed, and hence there exists an integer $j > k$ such that $\tau(j) = a$. This implies that the sequence $\tau(1) \cdots \tau(n)$ contains the 3412-subsequence $k \ j \ \tau(k) \ a$.

Conversely, suppose that the Motzkin path $\Phi(\tau)$ is maximally labelled. Suppose now that τ contains the 3412-subsequence $a b c d$. Remark that, in this case, both the subsequences of excedances and deficiencies of τ must be decreasing. This implies that at least one among the integers a, b, c , and d must be a fixed point of τ . If a or b is fixed, then c and d are deficiencies in increasing order. Similarly, if c or d is fixed, then a and b are excedances. In both cases, we get a contradiction.

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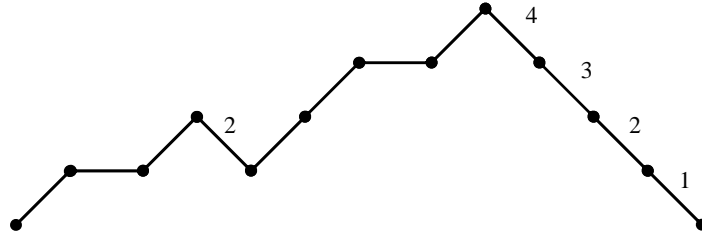


Figure 6: The labelled Motzkin path associated with the 3412 avoiding involution $\tau = 12\ 2\ 4\ 3\ 11\ 10\ 7\ 9\ 8\ 6\ 5\ 1$

The preceding result yields a bijection between the sets \mathcal{M}_n and $I_n(3412)$, that can be also found in [9].

In [5] the cardinality of $I_n(3412, \tau)$ for every τ in S_3 and S_4 , and for several $\tau \in S_5$, is determined. In the following, we exhibit a characterization for labelled Motzkin paths corresponding to involutions in $I_n(3412)$ that avoid either a pattern $\pi \in S_3$ or $\pi = 4321$. As a consequence, we derive some of the enumerative results contained in [5].

Theorem 8 *Let τ be an involution in $I_n(3412)$ associated with the labelled Motzkin path (M, μ) . Then:*

- i. τ avoids 132 if and only if M does not contain either HU or DU . Similarly, τ avoids 213 if and only if M does not contain either DH or DU ;
- ii. τ avoids 321 if and only if the irreducible components of M are either H or UD ;

- iii. τ avoids 312, and hence 231, if and only if the irreducible components of M are either of type $U^a H D^a$ or $U^a D^a$;
- iv. the set of paths corresponding to involutions in $I_n(3412)$ avoiding 123 can be constructed recursively as follows: either $M = UH^a DUH^b D$, with $a + b = n - 4$, or M is obtained from a path of the same kind by prepending U and appending D ;
- v. τ avoids 4321 if and only if the irreducible components of M are of type either $UH^a D$ or H .

Proof

- i. Let (M, μ) be a Motzkin path with the maximal labelling. Then:
 - 1. if the k -th step in M is horizontal and the $(k + 1)$ -th step is an up step, then $\tau(k) = k$ and $\tau(k + 1) = j > k + 1$. Hence, τ contains the 132-pattern $k \ j \ k + 1$;
 - 2. if the k -th step in M is a down step and the $(k + 1)$ -th step is an up step, then $\tau(k) = j < k$ and $\tau(k + 1) = m > k + 1$. Hence, τ contains the 132-pattern $j \ m \ k + 1$;

Suppose now that M contains neither HU nor DU . In this case, $M = U^h S$, where S is a suffix containing h down steps and $n - 2h$ horizontal steps in some order. Then, the one-line notation of τ has a prefix $e_1 \cdots e_h$ consisting of all the excedances in decreasing order and a suffix $\tau(h + 1) \cdots \tau(n)$ obtained by interlacing the sequence of deficiencies in decreasing order and the sequence of fixed points in increasing order. It is easy to verify that such a τ avoids 132, since each deficiency in τ must be less than every fixed point and every excedance.

- ii. As stated in Proposition 3, 321-avoiding involutions correspond to Motzkin paths whose irreducible components are either Dyck paths or horizontal steps, with the unitary labelling. Hence, an involution in $I_n(3412, 321)$ must correspond to a Motzkin path of height at most 1 whose irreducible components are H and UD , as desired.
- iii. This characterization has been proved in Proposition 4.

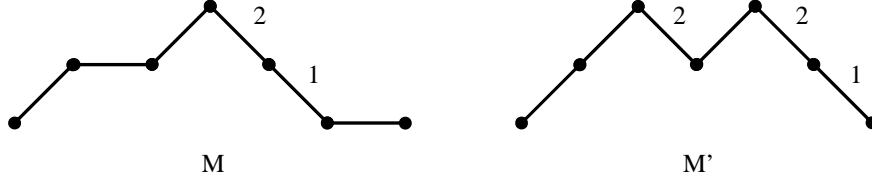


Figure 7: The labelled Motzkin paths M and M' corresponding to $\tau = 5 \mathbf{2} \mathbf{4} \mathbf{3} 1 6$ and $\tau' = 6 \mathbf{3} \mathbf{2} \mathbf{5} \mathbf{4} 1$, that contain 132.



Figure 8: The labelled Motzkin path M corresponding to $\tau = 1 \mathbf{3} \mathbf{2} \mathbf{5} \mathbf{4} \mathbf{6} \mathbf{8} \mathbf{7}$, that avoids 321.

- iv. Remark that an involution τ avoiding 123 must have at most 2 connected components. If τ has exactly 2 components, then there exists an integer k such that the one-line notation of τ is

$$k \ k-1 \ \cdots \ 1 \ n \ n-1 \ \cdots \ k+1.$$

Hence, the labelled Motzkin path $\Phi(\tau)$ has two irreducible components, of type either $U^a H D^a$ or $U^a D^a$, with the maximal labelling. It is easy to verify that each one of these Motzkin paths corresponds to an involution avoiding both 123 and 3412.

Consider now the case of a connected involution τ in $I_n(3412, 123)$, corresponding to an irreducible Motzkin path with the maximal labelling. In this case, we must have $\tau(1) = n$ and $\tau(n) = 1$. Hence, the involution $\tau' \in I_{n-2}$ obtained from τ by removing the symbols 1 and n and renormalizing the remaining symbols belongs to $I_{n-2}(3412, 123)$ if and only if τ belongs to $I_n(3412, 123)$.

- v. It is obvious that the height of a labelled Motzkin path corresponding to an involution avoiding both 4321 and 3412 can not exceed 1. Then, its irreducible components must be of type either $U H^a D$ or H .

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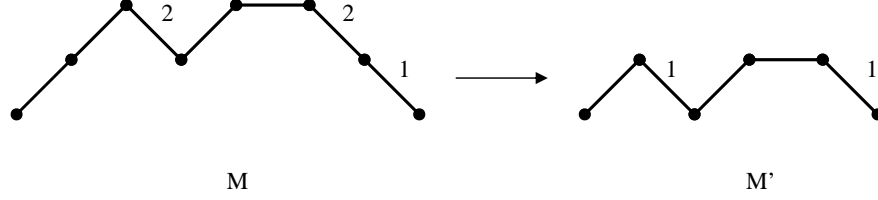


Figure 9: The Motzkin paths $M = \Phi(\tau)$ and $M' = \Phi(\tau')$, where $\tau = 7\ 3\ 2\ 6\ 5\ 4\ 1$ and $\tau' = 2\ 1\ 5\ 4\ 3$ avoid 123.

The following enumerative results stated in [5] can be now easily deduced from the preceding theorem:

Corollary 9 *We have:*

- i. $|I_n(3412, 132)| = |I_n(3412, 213)| = F_{n+1}$
where F_n is the n -th Fibonacci number;
- ii. $|I_n(3412, 321)| = F_{n+1}$;
- iii. $I_n(3412, 312) = I_n(3412, 231) = I_n(312)$ and $|I_n(312)| = 2^{n-1}$;
- iv. $|I_n(3412, 123)| = 1 + \lfloor \frac{n}{2} \rfloor \lceil \frac{n}{2} \rceil$
- v. $|I_n(3412, 4321)| = 2^{n-1}$.

◇

6 Fixed point free restricted involution

Consider the set DI_n of involutions without fixed points on n objects. It is evident that this set is nonempty if and only if $n = 2h$ is even. Involutions in DI_n correspond, via the bijection Φ , to labelled Dyck paths.

Theorem 2 allows us to characterize involutions in $DI_n(4321)$ as follows:

Theorem 10 *There is a bijection between the set $DI_{2h}(4321)$ and the set of \mathcal{D}_h of Dyck paths of length $2h$. Hence, $|DI_{2h}(4321)| = C_h$, where C_h is the h -th Catalan number.*

◇

It has been proved (see [3]) that also the cardinality of the set $DI_{2h}(321)$ is C_h . This fact is somehow surprising since $DI_n(321) \subseteq DI_n(4321)$. The described bijection yields an easy proof of the fact that these two sets coincide.

Proposition 11 *We have:*

$$DI_{2h}(321) = DI_{2h}(4321).$$

Proof The characterization of labelled Motzkin paths associated with involutions in $I_{2h}(321)$ given in Proposition 3 implies that the set $DI_{2h}(321)$ corresponds bijectively to the set of Dyck paths of semilength h with the unitary labelling. The assertion now follows from Theorem 10.

◇

The study of fixed point free involutions that avoid 4321 and a further pattern $\pi \in S_3$, with $\pi \neq 321$, has no interest. In fact, Theorem 5 implies that $|DI_n(4321, \pi)| \leq 2$ for every $n \in \mathbb{N}$.

Consider now the set $DI_{2h}(3412)$ of fixed point free involutions avoiding 3412. Also in this case, by Theorem 7, the set $DI_{2h}(3412)$ is in bijection with the set of Dyck paths of length $2h$, whose cardinality is C_h .

The characterization of the sets $DI_{2h}(3412, \pi)$, where π is one of the patterns considered in Theorem 8, is nontrivial only in the following cases:

Theorem 12 *For every integer n , we have:*

- a. $|DI_{2h}(3412, 123)| = 1 + \binom{h}{2};$
- b. $|DI_{2h}(3412, 312)| = 2^{h-1}.$

Proof

- a. Non-connected involutions in $DI_{2h}(3412, 123)$ correspond bijectively to Dyck paths of type $D = U^a D^a U^{h-a} D^{h-a}$, where $0 < a < h$. There are $h - 1$ involutions of this type. As seen in the proof of Theorem 8.iv, connected involutions in $DI_{2h}(3412, 123)$ are in bijection with involutions in $DI_{2h-2}(3412, 123)$. Hence, if we set $d_{2h} = |DI_{2h}(3412, 123)|$, we have:

$$d_{2h} = d_{2h-2} + h - 1.$$

Since $d_2 = 1$, we get the assertion.

- b. Remark that $DI_{2h}(3412, 312) = DI_{2h}(312)$. Involutions in $I_{2h}(312)$ without fixed points correspond bijectively to Dyck paths whose irreducible components are of type $U^a D^a$ (see Theorem 4). These paths are encoded by the compositions of $2h$ into even parts.

◇

7 Restricted centrosymmetric involutions

A *centrosymmetric involution* is an element $\tau \in I_n$ such that $\tau_{rc} = \tau$, namely, an involution such that $\tau(i) + \tau(n + 1 - i) = n + 1$, for every $1 \leq i \leq n$. Denote by CI_n the set of centrosymmetric involutions in I_n .

The study of pattern avoidance on the set of centrosymmetric involutions has been carried out by Egge [6] for every pattern $\pi \in S_3$. Moreover, Guibert and Pergola considered in [11] the case of vexillary centrosymmetric involutions, namely, centrosymmetric involutions avoiding 2143. In this section we specialize the previous results to the set CI_n .

In the case of centrosymmetric involutions, we have further equidistribution results among patterns. In fact, CI_n is closed under both reverse and complement operations, that was not the case for the set I_n . This implies that the patterns π , π_c , π_r , and π_{rc} are equidistributed over CI_n .

Theorem 1 implies that the Motzkin path associated with a centrosymmetric involution $\tau \in CI_n$ must be symmetric with respect to the vertical line $x = \frac{n}{2}$. The converse is false, in general. In fact, for instance, the involution associated with the symmetric labelled Motzkin path in Figure 10 is

$$\tau = 6 \ 2 \ 10 \ 4 \ 8 \ 1 \ 7 \ 5 \ 9 \ 3,$$

that is not centrosymmetric.

However, if a symmetric Motzkin path M is endowed with either the maximal or the unitary labelling, then the corresponding involution is centrosymmetric. In fact, consider for example the case of the unitary labelling, and suppose that there exists a symmetric Motzkin path M such that the corresponding involution $\tau \in I_n(4321)$ is not centrosymmetric, namely, $\tau_{rc} \neq \tau$. Then, also τ_{rc} avoids 4321, as seen in the proof of Theorem 5.i. This implies

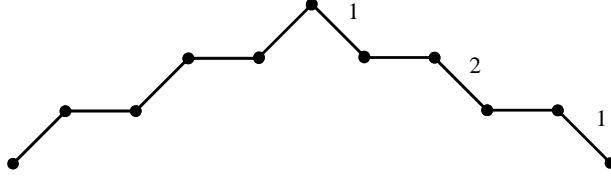


Figure 10: A symmetric labelled Motzkin path that is not associated with a centrosymmetric involution.

that τ_{rc} corresponds to the same Motzkin path with the same labelling, hence contradicting Theorem 2.

The case of the maximal labelling can be treated similarly, remarking that also the set $I_n(3412)$ is closed under the reverse-complement map.

The bijection between Motzkin paths and centrosymmetric involutions avoiding 4321 (or 3412) yields the following enumerative results:

Theorem 13 *For every integer h ,*

$$\begin{aligned} |CI_{2h+1}(4321)| &= |CI_{2h+1}(3412)| = |CI_{2h}(4321)| = \\ &= |CI_{2h}(3412)| = \sum_{i=0}^h \frac{h!}{(h-i)! \lfloor \frac{i}{2} \rfloor! \lceil \frac{i}{2} \rceil!}. \end{aligned} \quad (1)$$

Proof First of all, remark that a symmetric Motzkin path of length $2h+1$ can be uniquely obtained by adding a horizontal step in the middle position of a symmetric Motzkin path of length $2h$. Hence, we can restrict our attention to the even case. A symmetric Motzkin path of length $2h$ is completely determined by its first h steps. This means that we only need to count Motzkin prefixes of length h , whose formula can be found, for example, in [11].

◇

The following double restriction results can be derived from the analogous theorems for the set of all involutions:

Theorem 14 *We have:*

- i. $|CI_n(4321, 132)| = \lfloor \frac{n}{2} \rfloor$;
- ii. $CI_n(4321, 321) = CI_n(321)$. Moreover, we have:

$$|CI_{2h}(4321, 321)| = 2^h$$

$$|CI_{2h+1}(4321, 321)| = \binom{h}{\lfloor \frac{h}{2} \rfloor};$$
- iii. $|CI_n(4321, 312)| = t_{\lfloor \frac{n}{2} \rfloor + 1} + t_{\lfloor \frac{n}{2} \rfloor + 2}$,
 where t_k denotes the k -th tribonacci number;
- iv. $|CI_n(3412, 132)| = \lfloor \frac{n}{2} \rfloor$;
- v. $|CI_{2h}(3412, 321)| = F_{h+2}$ $|CI_{2h+1}(3412, 321)| = F_{h+1}$;
- vi. $CI_n(3412, 312) = CI_n(312)$ $|CI_n(3412, 312)| = 2^{\lfloor \frac{n}{2} \rfloor}$;
- vii. $|CI_{2h}(3412, 123)| = h + 1$ $|CI_{2h+1}(3412, 123)| = 1$;
- viii. $|CI_n(3412, 4321)| = 2^{\lfloor \frac{n}{2} \rfloor}$.

Proof

- i. as stated in Theorem 5.i, involutions avoiding both 4321 and 132 are in bijection with Motzkin paths of the kind $M = U^a H^b D^a H^c$. Among these, only the $\lfloor \frac{n}{2} \rfloor$ with $c = 0$ are symmetric.
- ii. See [6].
- iii. Involutions in $CI_n(4321, 312)$ correspond bijectively to symmetric Motzkin paths whose irreducible components are either H or UD or UHD . When n is odd, a path of this kind can be uniquely constructed by adding a horizontal step at the middle position of a Motzkin path of the same kind of length $n - 1$. Hence, we consider only the even case $n = 2h$. The first half of a Motzkin path of this type is either a (possibly non symmetric) Motzkin path of this type of length h or a Motzkin path of the prescribed type of length $h - 1$ followed by an up step. This completes the proof.
- iv. In this case, by Theorem 8.i, involutions avoiding both 3412 and 132 are in bijection with Motzkin paths of the kind $M = U^a S$. Among them, only the $\lfloor \frac{n}{2} \rfloor$ paths of type $M = U^a H^b D^a$ are symmetric.

- v. Involutions in $CI_n(3412, 321)$ correspond bijectively to symmetric Motzkin paths whose irreducible components are either H or UD . Consider first the even case $n = 2h$. The left half of a symmetric Motzkin path of the prescribed type can be either a (possibly non symmetric) Motzkin path of the same type of length h , or a Motzkin path of the same type of length $h-1$, followed by an up step U . Hence, the number of these paths is $F_{h+1} + F_h = F_{h+2}$, by Theorem 9.ii.

On the other hand, if $n = 2h + 1$, Motzkin paths of the prescribed type must be decomposable as follows

$$M = NHN,$$

where N is a Motzkin path whose connected components are either H or UD . Hence, by Theorem 9.ii, the number of these paths is F_{h+1} .

- vi. See [6].

- vii. Consider first the odd case. It is easily seen that all involutions in $CI_{2h+1}(3412, 123)$ are connected. Hence, as seen in the proof of Theorem 9.iv, a Motzkin path associated with an involution in $CI_{2h+1}(3412, 123)$ can be uniquely obtained by prepending U and appending D to a Motzkin path associated with an involution in $CI_{2h-1}(3412, 123)$. Since $CI_1(3412, 123)$ contains only one element, we have $|CI_{2h+1}(3412, 123)| = 1$.

If $n = 2h$, all involutions in $CI_{2h}(3412, 123)$ are connected, except for the involution corresponding to either $U^a HD^a U^a HD^a$ or $U^a D^a U^a D^a$, where $a = \lfloor \frac{h}{2} \rfloor$. Connected involutions in $CI_{2h}(3412, 123)$ correspond bijectively to the elements of $CI_{2h-2}(3412, 123)$, as remarked above. This implies that, setting $c_h = |CI_{2h}(3412, 123)|$, we have $c_h = c_{h-1} + 1$. Since $c_1 = 1$, we get the assertion.

- viii. The elements in $CI_n(3412, 4321)$ correspond to symmetric Motzkin paths of height at most 1. A path M of this type is completely determined by the prefix $p(M)$ consisting of the $\lfloor \frac{n}{2} \rfloor$ leftmost steps of M . Fix a non negative integer $h \leq \lfloor \frac{n}{2} \rfloor$. Motzkin prefixes whose rightmost point on the x -axis is $(h, 0)$ correspond bijectively to Motzkin paths of length h and height 1. In fact, such a prefix uniquely decomposes into

$p(M) = M'UH^b$, where $b = \lfloor \frac{n}{2} \rfloor - h - 1$. Hence, we have:

$$|CI_n(3412, 4321)| = 1 + \sum_{h=1}^{\lfloor \frac{n}{2} \rfloor} 2^{h-1} = 2^{\lfloor \frac{n}{2} \rfloor}.$$

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